Discrete symmetries and spectral statistics

1997 J. Phys. A: Math. Gen. 30 L177

(http://iopscience.iop.org/0305-4470/30/7/006)

J. Phys. A: Math. Gen. **30** (1997) L177–L181. Printed in the UK PII: S0305-4470(97)79755-5

LETTER TO THE EDITOR

Discrete symmetries and spectral statistics

J P Keating and J M Robbins

School of Mathematics, University of Bristol, Bristol BS8 1TW, UK
BRIMS, Hewlett-Packard Laboratories, Filton Road, Stoke Gifford, Bristol BS12 6QZ, UK

Received 21 November 1996

Abstract. We calculate the 2-point spectral statistics associated with a given irreducible representation (i.e. symmetry class) for time-reversal invariant systems possessing discrete symmetries using semiclassical periodic orbit theory. When the representation in question is real or pseudo-real, our results conform to those of the Gaussian orthogonal ensemble (GOE) of random matrices. When it is complex, we find instead Gaussian unitary ensemble (GUE) behaviour. This provides a direct semiclassical explanation for the recent observation by Leyvraz *et al* (1996) of GUE correlations in the desymmetrized spectra of certain symmetric billiards in the absence of any time-reversal invariance breaking (e.g. magnetic) fields.

The quantum energy levels of time-reversal invariant systems are expected typically to be correlated like the eigenvalues of matrices in the Gaussian orthogonal ensemble (GOE) of random matrix theory (RMT) (Bohigas *et al* 1984, Berry 1987, Bohigas 1991, Mehta 1991). To obtain Gaussian unitary ensemble (GUE) statistics, it is usually assumed that one must break time-reversal invariance; for example, by adding a magnetic field (Berry and Robnik 1986). However, recent numerical computations (Leyvraz *et al* 1996, hereafter referred to as LSS), inspired by ideas concerning structural invariance and the unitary representations of canonical transformations (Leyvraz and Seligman 1992), have shown that it is possible to find GUE statistics in the spectra associated with particular symmetry classes of certain symmetric, time-reversal invariant systems.

Our purpose here is to support this conclusion by presenting a general semiclassical calculation of the 2-point spectral statistics for symmetric systems using the link with classical periodic orbits provided by the trace formula (Gutzwiller 1971). Specifically, we shall focus on the dependence of the form factor $K(T)$ (the Fourier transform of the 2-point autocorrelation function of the density of states) for a given symmetry class on the properties of the associated irreducible representation of the symmetry group. Our approach is based directly on the semiclassical theory of spectral rigidity developed by Hannay and Ozorio de Almeida (1984) and Berry (1985). The result we find is that the levels associated with real or pseudo-real representations exhibit GOE correlations, whilst those associated with complex representations are GUE distributed. This agrees with the predictions made in LSS. To illustrate the general method, the example studied by LSS, namely a billiard with threefold rotational symmetry, will be treated explicitly.

If a system possesses discrete symmetries then the density of levels E_n, $d(E) = \sum_n \delta(E - E_n)$, may be split into a sum over the spectra associated with each of the symmetry classes α (i.e. the irreducible representations of the associated symmetry group

G): $d(E) = \sum_\alpha d_\alpha(E)$. Our aim is to calculate the behaviour of the form factor, defined for a given such desymmetrized spectrum by

$$K_\alpha(T) = \frac{1}{n_\alpha \bar{d}_\alpha} \int_{-\infty}^{\infty} \exp(\mathrm{i}xT/\hbar) \langle d_\alpha(E + x/2) d_\alpha(E - x/2)\rangle_E \, \mathrm{d}x - \frac{2\pi\hbar\bar{d}_\alpha}{n_\alpha}\delta(T) \tag{1}$$

where $\langle \ldots \rangle_E$ denotes an energy average, $\bar{d}(E) = \langle d_\alpha(E)\rangle_E$ is the corresponding mean density (Lauritzen and Whelan 1995), n_α is the symmetry-related level degeneracy, and the normalization is chosen so that $K_\alpha(T) \to 1$ as $T \to \infty$. We do this by substituting in the semiclassical trace formula for d_α (Robbins 1989, Lauritzen 1991, Cvitanovic and Eckhardt 1993)

$$d_\alpha(E) \approx \bar{d}_\alpha(E) + \frac{r_\alpha}{\pi\hbar} \mathrm{Re} \sum_j \chi_\alpha(g_j) A_j \exp(\mathrm{i}S_j/\hbar). \tag{2}$$

Here A_j and S_j are the usual (Gutzwiller 1971) amplitude and phase for orbits in the fundamental domain (S_j is here defined so as to include any Maslov indices), χ_α is the character and r_α the dimension of the irreducible representation, and g_j is the group element of G that relates initial and final points of the unfolded orbit in the full domain. The result is that for $T > 0$

$$K_\alpha(T) \approx \frac{1}{T_{\mathrm{H}}^{(\alpha)}} \sum_j \sum_k \langle \chi_\alpha(g_j)\chi_\alpha^*(g_k) A_j A_k \exp(\mathrm{i}\{S_j - S_k\}/\hbar)\delta(T - \tfrac{1}{2}(T_j + T_k))\rangle_E \tag{3}$$

where

$$T_{\mathrm{H}}^{(\alpha)} = \frac{2\pi\hbar\bar{d}\alpha}{n_\alpha} \tag{4}$$

is the appropriate Heisenberg time (i.e. the time conjugate to the mean level separation $n_\alpha/\bar{d}_\alpha$) and we have used the fact that $n_\alpha = r_\alpha$. The expression (3) is the direct analogue of the semiclassical formula derived by Berry (1985).

Before proceeding, it may be helpful to focus on the particular example studied in LSS: a billiard with threefold rotational symmetry but no reflection symmetry (i.e. $G = C_3$). In this case, the angular momentum modulo 3, which we denote by l, is a good quantum number and can be used to label the states. The eigenfunctions then transform according to $\psi_l \to \exp(2\pi\mathrm{i}l/3)\psi_l$ under rotation by 120°. For states with $l = 0$, one can always find a basis in which the eigenfunctions are real. When $l = \pm 1$, however, this is not possible, and it is for this reason that one might expect GUE spectral statistics. This is not inconsistent with the fact that the Hamiltonian is time-reversal invariant, because each $l = 1$ state has a time-reversed (i.e. complex conjugated) partner with $l = -1$ with which it is degenerate; hence linear combinations can always be constructed that are real.

For such a system the elements of the symmetry group are $C_3 = \{\omega^j\}_{j=0,1,2}$, where $\omega = \exp(2\pi\mathrm{i}/3)$, the associated characters are $\chi_l(\omega^j) = \omega^{lj}$, and $n_l = 1$. The periodic orbits in the fundamental domain (in this case, the third of the billiard lying between the rays $\theta = 0$ and $\theta = 2\pi/3$, with corresponding points on these rays identified) thus appear in the trace formula (2) with a phase factor $\chi_l(g_j) = \exp(2\pi\mathrm{i}lw_j/3)$, where w_j is the winding number of the jth orbit about the origin in the fundamental domain. This makes the semiclassical origins of the level statistics clear: when $l = 0$ the orbit contributions are time-reversal symmetric and so the statistics are GOE; but when $l \neq 0$ these contributions are the same as if there were an Aharonov–Bohm magnetic flux line of quantum flux strength $\phi = \frac{1}{3}$ situated at the origin of the desymmetrized billiard. It is this apparent magnetic flux, introduced by the desymmetrization into different angular momentum states (irreducible representations), that may be viewed as causing the transition to GUE statistics in the same way as in real Aharonov–Bohm billiards (Berry and Robnik 1986).

We now return to the calculation for the case of a general discrete symmetry to establish the conditions under which GOE and GUE statistics may be expected to describe the spectral fluctuations for a given irreducible representation. Our next step is to consider the diagonal ($S_j = S_k$) terms in the sum over orbit pairs in (3). There are two main reasons for doing this. First, these terms are known to dominate the off-diagonal contributions for $T \ll T_{\mathrm{H}}^{(\alpha)}$ (Berry 1985), and so to determine the spectral correlations over ranges much larger than the mean level separation $n_\alpha/\bar{d}_\alpha$ (in which range the unresummed semiclassical approximation (3) is expected to be valid (Keating 1994)). Second, the bootstrapping method (Bogomolny and Keating 1996) to extend the periodic orbit calculation of $K(T)$ to all values of T is based on a leading-order evaluation of the off-diagonal contributions directly in terms of the diagonal ones. The diagonal terms thus contain, to a first approximation, all long-range 2-point statistical information about the spectral correlations.

The diagonal pairs generically fall into two categories. In the first, the labels j and k refer to the same orbit, and so $A_j = A_k$, $S_j = S_k$, $T_j = T_k$ and $\chi_\alpha(g_j) = \chi_\alpha(g_k)$. In the second, the orbit labelled k is the time-reverse of the orbit labelled j. We represent this by writing $k = j^*$. Again, $A_j = A_{j^*}$, $S_j = S_{j^*}$ and $T_j = T_{j^*}$, but for point symmetries $\chi_\alpha^*(g_{j^*}) = \chi_\alpha^*(g_j^{-1}) = \chi_\alpha(g_j)$, where the first equality holds because in configuration space the time-reversed orbit is just the original one oppositely traversed. The diagonal terms may thus be written

$$K_\alpha^{(\mathrm{diag})}(T) = \frac{1}{T_{\mathrm{H}}^{(\alpha)}} \sum_j \langle A_j^2 \{ |\chi_\alpha(g_j)|^2 + (\chi_\alpha(g_j))^2 \} \delta(T - T_j) \rangle_E. \tag{5}$$

The amplitudes A_j^2 vary much more slowly than the characters $\chi_\alpha(g_j)$ with changes in the periods T_j. Indeed to a first approximation the group elements g_j, and hence the character values, are not correlated with the periods, because unfolded orbits with almost identical periods can have their initial and final points related in completely different ways. Thus in evaluating the orbit sum it is legitimate to first average over the character values. We do this by replacing $|\chi_\alpha(g_j)|^2$ and $(\chi_\alpha(g_j))^2$ by their group averages, defined by

$$\langle f(g) \rangle_G = \frac{1}{|G|} \sum_{g \in G} f(g). \tag{6}$$

This assumes that the group elements g_j for orbits with $T_j \approx T$ are uniformly distributed in G. For chaotic systems this is a natural approximation that is expected to hold in the limit as $T \to \infty$; that is, if T is measured in units of the Heisenberg time, it is valid in the semiclassical limit for any fixed dimensionless time. The analogue for Aharonov–Bohm billiards with a rational flux $\phi = n/m$ is that the winding numbers of orbits around the flux line may be taken to be uniformly distributed modulo m. In fact, these winding numbers are usually assumed to be Gaussian distributed with a variance proportional to $\sqrt{T}$ (Berry and Robnik 1986), and so when $\sqrt{T} \gg m$ the distribution is indeed approximately uniform, this approximation clearly improving as $T \to \infty$. Also, as has already been noted, for threefold rotationally symmetric billiards the characters appearing in the form factor may be interpreted as being determined by the winding numbers around an apparent flux line with $\phi = \frac{1}{3}$ at the origin of the fundamental domain.

To calculate the group averages we first use the well known orthonormality of the characters, which implies that

$$\langle |\chi_\alpha(g)|^2 \rangle_G = 1. \tag{7}$$

Second, we have that if $\chi_\alpha(g)$ is a character of an irreducible representation of G then its complex conjugate is as well, there being two possibilities: either $\chi^*_\alpha(g) = \chi_\alpha(g)$, in which case the representation is said to be real or pseudo-real†; or $\chi^*_\alpha(g) = \chi_\beta(g) \neq \chi_\alpha(g)$, in which case it is said to be complex. We may then use the general formula (see, for example, Hamermesh 1962)

$$\langle(\chi_\alpha(g))^2\rangle \equiv \beta - 1 = \begin{cases} 1 & \text{if } \alpha \text{ is real or pseudo-real} \\ 0 & \text{if } \alpha \text{ is complex.} \end{cases} \tag{8}$$

Substituting (7) and (8) into (5) thus gives

$$K_\alpha^{(\mathrm{diag})}(T) = \frac{\beta}{T_{\mathrm{H}}^{(\alpha)}} \sum_j \langle A_j^2 \delta(T - T_j)\rangle_E. \tag{9}$$

Finally, using the sum rule of Hannay and Ozorio de Almeida (1984) to evaluate the sum over periodic orbits, we find that

$$K_\alpha^{(\mathrm{diag})}(T) = \beta \frac{T}{T_{\mathrm{H}}^{(\alpha)}}. \tag{10}$$

When $\beta = 1$ this result coincides with the diagonal contribution to $K(T)$ for GUE-correlated spectra, as calculated by Berry (1985) for non-time-reversal-invariant systems: when $\beta = 2$ it coincides with the corresponding GOE expression. The implication of the above semiclassical calculation is therefore that spectra associated with real or pseudo-real irreducible representations should be GOE correlated, whilst those associated with complex representations should exhibit GUE statistics. This agrees precisely with the behaviour conjectured in LSS on the basis of previous studies of the unitary representations of canonical transformations (Leyvraz and Seligman 1992). In particular, for the case of threefold rotationally symmetric systems, the irreducible representation corresponding to $\alpha = 0$ is real, but the other two representations are complex. Our results thus provide a direct semiclassical periodic orbit theory for the numerical computations discussed in LSS.

We are grateful to Professors Leyvraz, Schmit and Seligman for making their results available to us prior to publication.

References

Berry M V 1985 *Proc. R. Soc. Lond.* A **400** 229–51
——1985 *Proc. R. Soc. Lond.* A **413** 183–98
Berry M V and Robnik M 1986 *J. Phys. A: Math. Gen.* **19** 649–68
Bogomolny E B and Keating J P 1996 *Phys. Rev. Lett.* **77** 1472–5
Bohigas O 1991 Random matrix theories and chaotic dynamics *Les Houches Lecture Series 52* ed M-J Giannoni, A Voros and J Zinn-Justin (Amsterdam: North-Holland) pp 89–199
Bohigas O, Giannoni M-J and Schmit C 1984 *Phys. Rev. Lett.* **52** 1–4
Cvitanovic P and Eckhardt B 1993 *Nonlinearity* **6** 277–311
Gutzwiller M C 1971 *J. Math. Phys.* **12** 343–58
Hamermesh M 1962 *Group Theory* (Reading, MA: Addison-Wesley)
Hannay J H and Ozorio de Almeida A M 1984 *J. Phys. A: Math. Gen.* **17** 3429–40
Keating J P 1994 *J. Phys. A: Math. Gen.* **27** 6605–15
Lauritzen B 1991 *Phys. Rev.* A **43** 603–6
Lauritzen B and Whelan N D 1995 *Ann. Phys.* **244** 112–35

† While both real and pseudo-real representations have real characters, the matrix representations of the former can be chosen to be real, whereas for the latter they cannot. An example is provided by the irreducible representations of the rotation group, which are real for integer spin and pseudo-real for half-integer spin.

Leyvraz F, Schmit C and Seligman T H 1996 *J. Phys. A: Math. Gen.* **29** L575–80
Leyvraz F and Seligman T H 1992 *Phys. Lett.* **168A** 348–52
Mehta M L 1991 *Random Matrices and the Statistical Theory of Energy Levels* 2nd edn (New York: Academic)
Robbins J M 1989 *Phys. Rev.* A **40** 2128–36